ON THE DERIVATION OF THE FREQUENCY FUNCTION OF SPACE VELOCITIES OF THE STARS FROM THE OBSERVED RADIAL VELOCITIES[1]

One of the important problems of stellar statistics is the derivation of the frequency function of the space velocities of stars of various spectral types and of different absolute magnitudes. The direct solution of this problem requires the knowledge of the space velocities of a great number of stars. The derivation of the space velocity of a given star is possible only if three different quantities are measured: the radial velocity, the proper motion and the parallax. These quantities are measurable with different relative degrees of accuracy and are exposed to systematic errors of quite different kinds. For some important groups of stars (for example B–type stars) we have very few reliable individual parallaxes. The number of stars with reliable parallaxes is generally small, and among them the radial velocities are known only for a fraction.

Therefore several writers have made the attempted to obtain some knowledge about the distribution law of space velocities from the radial velocities alone. However, in every case some more or less arbitrary form of this law was assumed, and the problem was restricted to finding the numerical values of some parameters entering in this form of distribution law. In the majority of cases these constants are the elements of the velocity ellipsoids.

Owing to the relative uniformity of the catalogues of the radial velocities, the results of the statistical investigations based on them are almost free from the influence of systematic errors. It seems desirable, then, to try to solve the problem of derivation of the frequency function of space velocities from the distribution of radial velocities without making any hypothesis about the form of this function.

So far as it is known to the writer, this problem not only remains yet unsolved, but is not even discussed in any detail. The purpose of the present paper is to derive a general formula for the frequency function of space velocities from the distribution of radial velocities.

For the frequency function of the space velocities we will derive and solve an integral equation. In this equation the observed frequency function of the radial velocities for the different parts of the sky enters as the known function.

The fundamental Assumption. We shall assume that the different elementary volumes of space in our neighbourhood have practically identical frequency functions of the space velocities. In reality, for rare types of stars (for example Cepheids), it is necessary to consider also the distant stars, since the number of stars of such types in our neighbourhood is very small. In such cases some corrections for the difference between the frequency functions in various parts of the galaxy are required. These corrections are beyond the scope of the present paper. We suppose that radial velocities of a sufficiently large number of near stars situated in different parts of the sky are given, and our aim is to derive the frequency function of the space velocities.

[1]Comunicated by Sir Arthur Eddington

We shall consider first the two–dimensional problem. It is of special interest, since some types of stars are strongly concentrated near the galactic plane and the z–components of their velocities are small.

The Two–dimensional Problem. The stars are distributed over a plane and we are situated in the same plane. We measure for each star the radial velocity V and its apparent position or azimuth. In the case of the stars of high galactic concentration the role of such azimuth is played by the galactic longitude. Let $f(V,\alpha)\,dV\,d\alpha$ be the number of the observed stars with azimuths between α and $\alpha + d\alpha$ and with radial velocities between V and $V + dV$. The function $f(V,\alpha)$ can be obtained from the lists of the radial velocities of stars. If, further, $n(\alpha)\,d\alpha$ is the total number of the observed stars in the directions between α and $\alpha + d\alpha$, we have

$$n(\alpha) = \int_{-\infty}^{\infty} f(V,\alpha)\,dV.$$

Let $\phi(\xi,\eta)$ be the unknown frequency function of the velocities, i.e. $\phi(\xi,\eta)\,d\xi\,d\eta$ is the relative number of stars for which the velocity components fall within the limits ξ and $\xi + d\xi$, η and $\eta + d\eta$. We have

$$\int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} \phi(\xi,\eta)\,d\xi = 1.$$

Among the observed stars with the azimuths within the interval $(\alpha, \alpha + d\alpha)$ we have $n(\alpha)\,d\alpha\,\phi(\xi,\eta)\,d\xi\,d\eta$ stars, which have the velocities within the element $d\xi\,d\eta$ of the "velocity plane" $\xi\eta$.

We may choose the direction of the ξ–axis to have the azimuth $\alpha = 0$. Then it is clear that all stars observed in the azimuth α for which the velocities lie within the strip S of $\xi\eta$ plane (see fig.1) have the radial velocities lying between V and $V + dV$. Therefore from $n(\alpha)\,d\alpha$ stars within the interval $(\alpha, \alpha + d\alpha)$,

$$n(\alpha)\,d\alpha \int_{(S)} \phi(\xi,\eta)\,d\xi\,d\eta$$

stars will have the radial velocities between V and $V + dV$. The integration is carried over the strip (S), perpendicular to the direction α and with the width dV.

On the other hand, we have denoted the number of such stars as

$$f(V,\alpha)\,dV\,d\alpha.$$

Therefore we have the equation

$$f(V,\alpha)\,dV = n(\alpha) \int_{(S)} \phi(\xi,\eta)\,d\xi\,d\eta. \tag{1}$$

Let us make a change of variables:

$$\xi' = \xi\cos\alpha + \eta\sin\alpha,$$
$$\eta' = -\xi\sin\alpha + \eta\cos\alpha.$$

It is clear that ξ' within the strip (S) varies between V and $V+dV$, and η' varies between $-\infty$ and ∞. Therefore

$$\int_{(S)} \phi(\xi,\eta)\, d\xi\, d\eta = \int_V^{V+dV} d\xi' \int_{-\infty}^{\infty} \phi(\xi'\cos\alpha - \eta'\sin\alpha, \xi'\sin\alpha + \eta'\cos\alpha)\, d\eta'.$$

We divide the equation (1) by $n(\alpha)$ and set

$$F(V,\alpha) = \frac{f(V,\alpha)}{n(\alpha)} = \frac{f(V,\alpha)}{\int f(V,\alpha)\, dV},$$

to obtain the equation

$$F(V,\alpha) = \int_{-\infty}^{\infty} \phi(V\cos\alpha - \eta'\sin\alpha, V\sin\alpha + \eta'\cos\alpha)\, d\eta'. \tag{2}$$

The left–hand side of this equation may be obtained from the counts of stars in the radial–velocities lists.

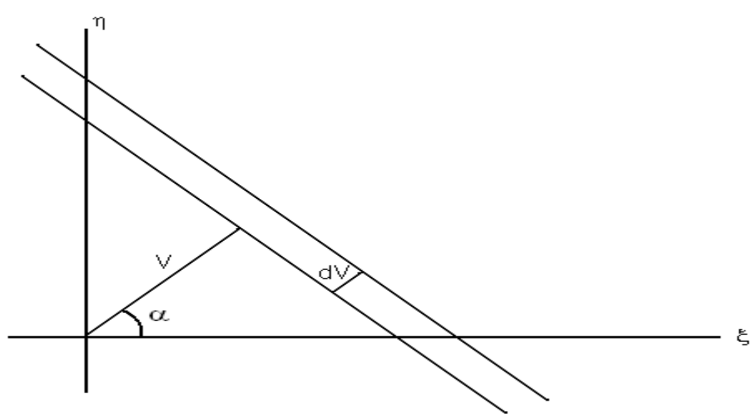

Fig.1

Returning to the old co–ordinates ξ and η, we may write the equation (2) in the form

$$F(V.\alpha) = \int_{(L)} \phi(\xi,\eta)\, ds, \tag{3}$$

where the integration is carried over the straight line (L) defined by

$$\xi\cos\alpha + \eta\sin\alpha = V,$$

and ds is the element of this line.

We have come to the following problem:

The value of the integral (3) for every straight line of the $\xi\eta$ *plane is given as the function of the parameters* V *and* α*, defining the straight line. The integrand* $\phi(\xi, \eta)$ *is to be found.*

The solution of this problem is comparatively simple. Let us introduce in both parts of (2) instead of V the expression

$$V = x \cos\alpha + y \sin\alpha + W, \tag{5}$$

where x, y and W are some arbitrary parameters. Then we can rewrite (2) as

$$F(x \cos\alpha + y \sin\alpha + W, \alpha) = \int_{-\infty}^{\infty} \phi(x \cos^2\alpha + y \sin\alpha \cos\alpha + W \cos\alpha - \eta' \sin\alpha,$$
$$x \cos\alpha \sin\alpha + y \sin^2\alpha + W \sin\alpha + \eta' \cos\alpha)\, d\eta'.$$

If we introduce the new variable of integration

$$\eta' = U - x \sin\alpha + y \cos\alpha, \tag{6}$$

our equation takes the simple form

$$\begin{aligned} &F(x \cos\alpha + y \sin\alpha + W, \alpha) = \\ &= \int_{-\infty}^{\infty} \phi(x + W \cos\alpha - U \sin\alpha, y + W \sin\alpha + U \cos\alpha)\, dU. \end{aligned} \tag{7}$$

Multiplying both parts by $d\alpha$, integrating between 0 and 2π and changing the order of the integration on the right–hand side, we find

$$\begin{aligned} &\int_0^{2\pi} F(x \cos\alpha + y \sin\alpha + W, \alpha)\, d\alpha = \\ &= \int_{-\infty}^{\infty} dU \int_0^{2\pi} \phi(x + W \cos\alpha - U \sin\alpha, y + W \sin\alpha + U \cos\alpha)\, d\alpha. \end{aligned} \tag{8}$$

Now it is easy to see that the integral

$$\Phi = \int_0^{2\pi} \phi(x + W \cos\alpha - U \sin\alpha, y + W \sin\alpha + U \cos\alpha)\, d\alpha, \tag{9}$$

depends only on x, y and $\sqrt{W^2 + U^2}$. In fact, if we introduce in (9)

$$\begin{aligned} W &= G \cos\beta, \\ U &= G \sin\beta, \end{aligned} \tag{10}$$

we obtain

$$\Phi = \int_0^{2\pi} \phi[x + G \cos(\alpha + \beta), y + G \sin(\alpha + \beta)]\, d\alpha,$$

and it is obvious that this integral depends only on x, y and $G = \sqrt{W^2 + U^2}$ and is independent of $\beta \equiv \arctan \frac{U}{W}$.
Therefore we may write simply:

$$\Phi(x, y, G) = \int_0^{2\pi} \phi(x + G\cos\alpha, y + G\sin\alpha)\, d\alpha, \tag{11}$$

and

$$\int_0^{2\pi} F(x\cos\alpha + y\sin\alpha + W, \alpha)\, d\alpha = \int_{-\infty}^{\infty} \Phi(x, y, G)\, dU. \tag{12}$$

However,

$$dU = \frac{G\, dG}{\sqrt{G^2 - W^2}},$$

and we may rewrite (12) in the form

$$\int_0^{2\pi} F(x\cos\alpha + y\sin\alpha + W, \alpha)\, d\alpha = 2\int_W^{\infty} \Phi(x, y, G)\, \frac{G\, dG}{\sqrt{G^2 - W^2}}. \tag{13}$$

This equation is an integral equation of Abel's type for the function $\Phi(x, y, G)$, and its solution is given by

$$\Phi(x, y, G) = -\frac{1}{\pi}\frac{1}{G}\frac{d}{dG}\int_G^{\infty} \frac{W\, dW}{\sqrt{W^2 - G^2}} \int_0^{2\pi} F(x\cos\alpha + y\sin\alpha + W, \alpha)\, d\alpha. \tag{14}$$

We have, according to (11),

$$\Phi(x, y, 0) = 2\pi\phi(x, y), \tag{15}$$

and we may rewrite (14) in the form

$$\phi(x, y) = -\frac{1}{2\pi^2} \lim_{G \to 0} \frac{1}{G}\frac{d}{dG}\int_G^{\infty} \overline{F}(x, y, W)\, \frac{W\, dW}{\sqrt{W^2 - G^2}}, \tag{16}$$

where the function

$$\overline{F}(x, y, W) = \int_0^{2\pi} F(x\cos\alpha + y\sin\alpha + W, \alpha)\, d\alpha \tag{17}$$

may be obtained from the observations. After some algebra we reduce (16) to

$$\phi(x, y) = -\frac{1}{2\pi^2}\int_0^{\infty} \frac{1}{W}\frac{d\overline{F}(x, y, W)}{dW}\, dW. \tag{18}$$

This equation is the solution of our problem. The numerical calculation of $\overline{F}(x, y, W)$, when $F(V, \alpha)$ is given, may be carried out without difficulty.

We have actually applied our formulae to the radial velocities of B–type stars observed in the galactic belt $\xi|b| < 20^\circ$, and the results of this application are in satisfactory agreement with the velocity distribution derived from the direct counts of the known space velocities.
The details of this application will be given elsewhere.

The Three–dimensional Problem. In the case of the three–dimensional problem we may derive from the catalogues the number of stars observed within the given solid angle $d\omega$ in the given direction, having the radial velocity confined within the limits V and $V + dV$. Let us denote this number by $f(V, l, b)\, d\omega$, where l and b are galactic longitude and latitude. If, further, $n(l, b)\, d\omega$ is the total number of observed stars in the same solid angle, we shall have

$$n(l, b) = \int f(V, l, b)\, dV. \tag{19}$$

We have the following relation between the frequency function of the space velocities $\phi(\xi, \eta, \zeta)$ and the observed function $f(V, l, b)$:

$$f(V, l, b)\, dV = n(l, b) \iiint_\Omega \phi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta, \tag{20}$$

where the integration is extended over the volume (Ω) between two parallel planes, which are perpendicular to the direction (l, b) and have the distances V and $V + dV$ from the origin.

Dividing (20) by $n(l, b)$, we bring this equation after some transformations to the form

$$F(V, l, b) = \iint_\Sigma \phi(\xi, \eta, \zeta)\, d\sigma, \tag{21}$$

where

$$F(V, l, b) = \frac{f(V, l, b)}{n(l, b)}, \tag{22}$$

and the integration is extended over the plane (Σ), perpendicular to the direction (l, b), at distance V from the origin.
The equation of this plane is:

$$\xi \cos l \cos b + \eta \sin l \cos b + \zeta \sin b = V. \tag{(Σ)}$$

If we introduce the polar co–ordinates ρ and θ in the plane (Σ) with origin at the point

$$\xi = V \cos l \cos b; \quad \eta = V \sin l \cos b; \quad \zeta = V \sin b,$$

we shall have for the points of this plane

$$\begin{aligned} \xi &= V \cos l \cos b + \rho(\alpha_1 \cos\theta + \beta_1 \sin\theta), \\ \eta &= V \sin l \cos b + \rho(\alpha_2 \cos\theta + \beta_2 \sin\theta), \\ \zeta &= V \sin b + \rho(\alpha_3 \cos\theta + \beta_3 \sin\theta), \end{aligned}$$

where the coefficients $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ satisfy the conditions

$$\alpha_1 \cos l \cos b + \alpha_2 \sin l \cos b + \alpha_3 \sin b = 0,$$

$$\beta_1 \cos l \cos b + \beta_2 \sin l \cos b + \beta_3 \sin b = 0,$$

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0,$$
$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \qquad \beta_1^2 + \beta_2^2 + \beta_3^2 = 1.$$

Now we can rewrite the equation (21) in these coordinates:

$$F(V,l,b) = \int_0^\infty \rho\, d\rho \int_0^{2\pi} \phi(V\cos l\cos b + \rho(\alpha_1\cos\theta + \beta_1\sin\theta),$$
$$V\sin l\cos b + \rho(\alpha_2\cos\theta + \beta_2\sin\theta), V\sin b + \rho(\alpha_3\cos\theta + \beta_3\sin\theta))\, d\theta.$$

Integrating over all directions and changing the order of the integration on the right–hand side, we obtain

$$\int F(V,l,b)\, d\omega = \int \Phi\, \rho\, d\rho, \tag{23}$$

where

$$\Phi = \int d\omega \int_0^{2\pi} \phi(V\cos l\cos b + \rho(\alpha_1\cos\theta + \beta_1\sin\theta),$$
$$V\sin l\cos b + \rho(\alpha_2\cos\theta + \beta_2\sin\theta), V\sin b + \rho(\alpha_3\cos\theta + \beta_3\sin\theta))\, d\theta.$$

In terms of parameters

$$V = G\gamma_1; \qquad \rho\cos\theta = G\gamma_2; \qquad \rho\sin\theta = G\gamma_3; \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,$$

the integral Φ takes the form

$$\Phi = \int d\theta \int \phi(G(\gamma_1\cos l\cos b + \gamma_2\alpha_1 + \gamma_3\beta_1),$$
$$G(\gamma_1\sin l\cos b + \gamma_2\alpha_2 + \gamma_3\beta_2), G(\gamma_1\sin b + \gamma_2\alpha_3 + \gamma_3\beta_3))\, d\omega,$$

and it may be shown that it depends only on G. We may write

$$\Phi(G) = 2\pi \int \phi(G\cos l\cos b, G\sin l\cos b, G\sin b)\, d\omega; \quad \phi(0) = 8\pi^2\phi(0,0,0). \tag{24}$$

We have now

$$G^2 = V^2 + \rho^2; \qquad \rho\, d\rho = G\, dG.$$

Therefore

$$\int F(V,l,b)\, d\omega = \int_V^\infty \Phi\, G\, dG \tag{25}$$

and

$$\Phi = -\frac{1}{V}\frac{d}{dV}\int F(V,l,b)\, d\omega. \tag{26}$$

Comparing (26) with (27) we find

$$\phi(0,0,0) = \frac{1}{8\pi^2}\Phi(0) = -\lim_{V\to 0}\frac{1}{V}\frac{d}{dV}\int F(V,l,b)\, d\omega.$$

Thus we may find $\phi(0,0,0)$. In the same way after some lengthy algebra we obtain

$$\phi(\xi,\eta,\zeta) = -\frac{1}{8\pi^2}\frac{1}{W}\frac{W}{dW}\int F(\xi\cos l\cos b + \eta\sin l\cos b + \zeta\sin b + W, l, b)\, d\omega.$$

This formula represents the solution of the three–dimensional problem.

Concluding Remarks. In our method it is supposed that the K–effect is absent. Actually, however, we may determine the K–term from the radial velocities and exclude it.

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