ON THE PROBLEM OF DIFFUSE REFLECTION OF LIGHT BY TURBID MEDIUM

(Presented by academician S. I. Vavilov 6. I. 1943)

The problem of diffuse reflection of light by a turbid medium i.e. by a medium, which both scatters and absorbs the light has been the subject of many investigations. The usual approach to this physical problem is based on the analysis of the radiation field within the medium and subsequent calculation using the received data, of the intensities of diffusely reflected radiation. However, by now this approach has not produced a complete solution of the problem.

In this note we propose another approach to the problem of diffuse reflection of light. It appears that this new method is much more effective than the old one, which led to an integral equation. The new method avoids any calculations connected with evaluation of quantities describing the radiation field inside the medium. Now we consider only what happens near its surface.

We take a medium, which consist of plane–parallel layers, each element of which has both scattering and absorbing ability. Let us suppose that the ratio $\dfrac{\lambda}{1-\lambda}$ of scattering coefficient to the absorbtion coefficient is a constant throughout the medium. Let us also suppose that the scattering indicatrix has everywhere a spherical form. This means that the light scattered by every element of volume is distributed uniformly in all direction.

Finaly, let us suppose that the medium is bounded on one side by a plane A, and on other side it stretches to infinity. This means that the optical depth is infinite. At the end of this paper we will consider also the case of a limited optical thickness.

Let us suppose that the plane A is illuminated by a bundle of parallel rays, which form some angle $\arccos\xi$ with the inner normal to A. Let the flux of this radiation through a unit of area perpendicular to the bundle be πS. Owing to scattering processes (multiple in the general case), certain intensities of diffuse light will flow from the medium in different directions. Of course the intensity I of the outgoing radiation will depend on the angle between the direction of the outgoing radiation and the direction normal to A. Let the cosinus of that angle be η. Thus I will depend on ξ and η, being at the same time proportional to S:

$$I(\eta,\xi) = r(\eta,\xi)\,S.$$

Our aim is to find the function $r(\eta,\xi)$.

Assume we attach to our infinite medium an additional layer of small optical thickness $\Delta\tau$ consisting of matter of the same optical properties. We think that the layer $\Delta\tau$ is bounded by two parallel planes, A and A'; thus A' forms the boundary of the new medium. The new composite medium will have the same ability of diffuse reflection, as the initial infinite medium, characterized by the same function $r(\eta,\xi)$.

This property of invariance with respect to addition of the layer $\Delta\tau$ we shall use to derive an equation for the function $r(\eta, \xi)$. Below we will neglect the quantities of the order of $\Delta\tau^2$ and higher.

As a result of addition of the layer $\Delta\tau$, now the light first penatrates the new boundary A' of the medium. On the old boundary A instead of the quantity πS the quantity $\pi S\left(1-\dfrac{\Delta\tau}{\xi}\right)$ of direct radiation will now fall. Therefore A will reflect the intensity $r(\eta,\xi)\,S\left(1-\dfrac{\Delta\tau}{\xi}\right)$. But during the passage of the layer $\Delta\tau$ it will diminish $\left(1-\dfrac{\Delta\tau}{\eta}\right)$ times. The corresponding contribution to I will be

$$r(\eta,\xi)\left(1-\frac{\Delta\tau}{\xi}\right)\left(1-\frac{\Delta\tau}{\eta}\right)S.$$

However, the layer $\Delta\tau$ will scatter an additional intensity in the direction η. This intensity consists of four parts:

1) The layer $\Delta\tau$ scatters a part of the direct beam in the direction of η. This part is equal to

$$\frac{\lambda}{4}\,\frac{\Delta\tau}{\eta}\,S.$$

2) A part of the radiation is scattered by the layer $\Delta\tau$ towards A and partly is reflected from A. This gives an additional intensity

$$\frac{\lambda}{2}\Delta\tau\,S\int_0^1 r(\eta,\zeta)\,\frac{d\zeta}{\zeta}.$$

3) The layer $\Delta\tau$ scatters the radiation reflected from A. The corresponding intensity is

$$\frac{\lambda}{2}\,\frac{\Delta\tau}{\eta}\,S\int_0^1 r(\zeta,\xi)\,d\zeta.$$

4) A part of radiation reflected from A is scattered back by the layer $\Delta\tau$ and again is partly reflected from A. As a result the intensity obtains an additional term

$$\lambda\Delta\tau S\int_0^1 r(\zeta,\xi)\,d\zeta\int_0^1 r(\eta,\zeta')\,\frac{d\zeta'}{\zeta'}.$$

Now owing to all these increases and decreases the intensity reflecting from A' remains equal to $r(\eta,\xi)\,S$. Therefore

$$\begin{aligned} r(\eta,\xi) = r(\eta,\xi)\left(1-\frac{\Delta\tau}{\xi}-\frac{\Delta\tau}{\eta}\right)+\frac{\lambda}{4}\frac{\Delta\tau}{\eta}+\frac{\lambda}{2}\frac{\Delta\tau}{\eta}\int_0^1 r(\zeta,\xi)\,d\zeta+ \\ +\frac{\lambda}{2}\Delta\tau\int_0^1 r(\eta,\zeta)\,\frac{d\zeta}{\zeta}+\lambda\Delta\tau\int_0^1 r(\zeta,\xi)\,d\zeta\int_0^1 r(\eta,\zeta')\,\frac{d\zeta'}{\zeta'} \end{aligned}$$

or

$$\left(\frac{1}{\eta}+\frac{1}{\xi}\right)\, r(\eta,\xi)=\frac{\lambda}{4}\left[\frac{1}{\eta}+2\int_0^1 r(\eta,\zeta)\,\frac{d\zeta}{\zeta}+\frac{2}{\eta}\int_0^1 r(\zeta,\xi)\,d\zeta+\right.$$
$$\left.+\,4\int_0^1 r(\eta,\zeta')\,\frac{d\zeta'}{\zeta'}\int_0^1 r(\zeta,\xi)\,d\zeta\right].$$

We introduce the function $R(\eta,\xi)$, defined by

$$R(\eta,\xi)=\frac{4\eta}{\lambda}\,r(\eta,\xi). \tag{1}$$

Then we obtain for $R(\eta,\xi)$ a functional equation

$$\begin{gathered}\left(\frac{1}{\eta}+\frac{1}{\xi}\right) R(\eta,\xi)=1+\frac{\lambda}{2}\int_0^1 R(\eta,\zeta)\,\frac{d\zeta}{\zeta}+\\ \frac{\lambda}{2}\int_0^1 R(\zeta,\xi)\,\frac{d\zeta}{\zeta}+\frac{\lambda^2}{4}\int_0^1 R(\eta,\zeta')\,\frac{d\zeta'}{\zeta'}\int_0^1 R(\zeta,\xi)\,\frac{d\zeta}{\zeta}.\end{gathered} \tag{2}$$

Evidently if a function $R(\eta,\xi)$ satisfies this equation so, then does the function $R(\xi,\eta)$. But since our physical problem should have only one solution, it is natural to try to find a symmetric solution, i.e. to assume that

$$R(\eta,\xi)=R(\xi,\eta). \tag{3}$$

But in this case the right hand side of (2) can be presented as a product

$$\left(\frac{1}{\eta}+\frac{1}{\xi}\right) R(\eta,\xi)=\left[1+\frac{\lambda}{2}\int_0^1 R(\eta,\zeta)\,\frac{d\zeta}{\zeta}\right]\left[1+\frac{\lambda}{2}\int_0^1 R(\xi,\zeta)\,\frac{d\zeta}{\zeta}\right]. \tag{4}$$

Let us denote

$$\varphi(\eta)=1+\frac{\lambda}{2}\int_0^1 R(\eta,\zeta)\,\frac{d\zeta}{\zeta}. \tag{5}$$

Then (4) implies the following structure of $R(\eta,\xi)$:

$$R(\eta,\xi)=\frac{\varphi(\eta)\,\varphi(\xi)}{\frac{1}{\eta}+\frac{1}{\xi}}. \tag{6}$$

Substituting (6) into (5), we obtain the equation for $\varphi(\eta)$

$$\varphi(\eta)=1+\frac{\lambda}{2}\eta\,\varphi(\eta)\int_0^1\frac{\varphi(\xi)\,d\xi}{\eta+\xi}. \tag{7}$$

Thus the function $R(\eta,\xi)$ which determines the law of diffuse reflection is expressed in the form (6), where $\varphi(\eta)$ is the solution of the functional equation (7).

Since in our physical problem we have always $\lambda\le 1$ the equation (7) is easily solved numerically by mean of sucsessive approximation, biginning with zero approximation $\varphi_0(\eta)=1$. In this way the problem of diffuse reflection from an infinitely deep medium can be completely solved.

One can also deduce the functional equations (2) and (7) not from the physical considerations as above but rather from the well known integral equation of the scattering theory. This was shown in our earlier paper (Russ. Astr. Journ., vol.19, no.5, 30, 1942).

The method exposed above can be applied not only to media of infinite depth but also to the layers of finite optical thickness τ, enclosed between two parallel planes A and B. In this case, however we consider not only the function of diffuse reflection $r(\eta, \xi)$ but also a function $s(\eta, \xi)$ which describes the diffuse transparency i.e. the part of light which enters via A and leaves the medium via B.

To use invariance, we add a thin layer $\Delta\tau$ on the side A and subtract the same layer on the side B.

Then is possible to present the functions $r(\eta, \xi)$ and $s(\eta, \xi)$ by means of two anxilary functions $\varphi(\eta)$ and $\psi(\eta)$, each of which depends on one variable only:

$$r(\eta, \xi) = \frac{\lambda}{4}\xi\, \frac{\varphi(\eta)\varphi(\xi) - \psi(\eta)\psi(\xi)}{\eta + \xi}$$

$$s(\eta, \xi) = \frac{\lambda}{4}\xi\, \frac{\psi(\eta)\varphi(\xi) - \varphi(\eta)\psi(\xi)}{\eta - \xi}$$

The functions $\varphi(\eta)$ and $\psi(\eta)$ must then satisfy the equations

$$\varphi(\eta) = 1 + \frac{\lambda}{2}\,\eta\varphi(\eta) \int_0^1 \frac{\varphi(\xi)\,d\xi}{\eta + \xi} - \frac{\lambda}{2}\eta\psi(\eta) \int_0^1 \frac{\psi(\xi)\,d\xi}{\eta + \xi}$$

$$\psi(\eta) = \exp\left(-\tau\,/\eta\right) + \frac{\lambda}{2}\eta \int_0^1 \frac{\psi(\eta)\varphi(\xi) - \varphi(\eta)\psi(\xi)}{\eta - \xi}\, d\xi.$$

The numerical solution of this system of two equations can be found also by successive approximations.

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Remark. The method of addition of layers has been applied later on by many authors. In particul it was applied five years later (in 1947) by S. Chandrasekhar, who of course was citing this paper of V. Ambartzumian. Some writers use for functions $\varphi(\xi)$ and $\psi(\xi)$ the term "Chandrasekhar's functions".