ON THE RADIATIVE EQUILIBRIUM OF A PLANETARY NEBULA

The beatiful works of Hubble, Bowen and Zanstra have solved at least qualitatively the problem of the origin of the nebular luminosity and of the nebular spectra. Carroll and Cillié have made an attempt to compute the relative intensities of the members of the Balmer series of hydrogen in nebular spectra, opening the door to theoretical interpretation of the modern spectrophotometric data. However the dynamic of nebulae as well as the nature and the origin of forces acting in them remain in darkness. It is now scarcely possible to answer these questions and to build up a complete theory of planetary nebulae.

Selective radiation pressure, owing to the specific nebular conditions, plays a very important part in the nebulae. Perhaps the radiation pressure is in this case larger than any other force. The computation of the radiation pressure is therefore a matter of considerable interest. This computation can be based on the analysis of the field of radiation. An approximate analysis can be done without the solution of other problems connected with planetary nebulae. The purpose of the present paper is the investigation of the radiative equilibrium of planetary nebulae. We concentrate on the planetary nebulae, though many results may be applied to the diffuse nebulae, as well as to the gaseous envelopes surrounding some stars with emission line spectrum (P Cygni and others). The method applied here was proposed by the writer in an earlier paper [1].

The Radiative Equilibrium of a Non–expanding Hydrogen Nebula

We will below consider the planetary nebula consisting of hydrogen only. There are some observational data indicating the expansion of planetary nebulae. (Owing to Doppler displacement, the frequencies of a spectral line in different parts of nebula are different.) This yields an appreciable change in the type of the radiative equilibrium.

But in some cases the expansion–velosity is so small that the frequency differences of a line for different parts of the nebula are smaller than its Doppler–broadening, caused by the thermal motion of atoms. The behaviour of such nebulae is the same as the behaviour of a non–expanding nebula, if we confine ourselves to the field of radiation and its interaction with nebular matter.

According to the theory of nebular luminosity developed by Zanstra, all, or at least a considerable part of quanta emitted by the central star, which have frequencies greater than ν_0 (the frequency of the limit of the Lyman series), are absorbed by the hydrogen atoms in the nebula. This circumstance requires that the optical thickness of the nebula τ_1 for these frequencies should be larger than unity or at least not much smaller than unity.

Following Zanstra's line of argument we will show that in the place of each absorbed quantum having the frequency greater than ν_0 there is a certain probability p of re–emission in the same frequency and the probability $1-p$ of the re–emission in the line $\mathbf{L}_\alpha$ (first line of the Lyman series of hydrogen). For simplicity, we will call the radiation beyond the head of the Lyman series briefly "ultra–violet radiation" and the corresponding quanta "ultra–violet quanta".

We consider the possible transformations of an ultra–violet quantum, which is emitted from the surface of the central star. It will be absorbed by the nebular envelope, and the absorption will be accompanied by the ionisation of a hydrogen atom. After an interval of time the freed electron is captured again by a proton. There are two possibilities at this capture: I, the electron jumps immediately into the deepest level 1S (first level); and II, the electron jumps into one of the excited levels.

In Case I, a new ultra-violet quantum is emitted and the initial state is restored. In Case II, the electron makes a chain of transitions, the last link of which will be a transition into the first level. The dilution of radiation is so great, and the density of matter so low, that the interruption of these transitions is very improbable. The last transition into the normal state is accompanied by the emission of a quantum of the Lyman series. There are again two possibilities. (a) An $\mathbf{L}_\alpha$–quantum is emitted.

However, Zanstra's theory requires that the optical thickness of the nebula in ultra–violet light be at least of the order of unity. But the coefficient of line absorption in the Lyman series is some thousand times larger than the absorption coefficient beyond the head of the series. The emitted $\mathbf{L}_\alpha$–quantum, therefore, will be absorbed by a hydrogen atom in the normal state. This atom passes into the second level, and then, owing to the absence of external perturbations during its short lifetime, turns back to the first level, emitting again an $\mathbf{L}_\alpha$–quantum. Thus the $\mathbf{L}_\alpha$–quantum remains unchanged, and we may say that it suffers only the scattering processes. These processes may be repeated many times until the quantum reaches the outer boundary of the nebula and escapes. (b) A quantum of some other line of the Lyman series is emitted. For simplicity we assume that it is an $\mathbf{L}_\beta$–quantum. In this line the optical thickness of the nebula is also very large, and the emitted quantum will be absorbed. This absorption is accompanied by the transition of an atom from the first to the third level. The atom in the third level has two possibilities: it either makes the transition of the type $3 \longrightarrow 2 \longrightarrow 1$, emitting the quanta $\mathbf{H}_\alpha$ and $\mathbf{L}_\alpha$ successively, or it passes immediately into the first level, emitting again $\mathbf{L}_\beta$. In the first case the final product is an $\mathbf{L}_\alpha$–quantum. Its further fate is described above. In the second case the quantum $\mathbf{L}_\beta$ will be absorbed, and thus there exists a finite probability of creation of a quantum $\mathbf{L}_\alpha$. After many absorptions and re–emissions the probability of the survival of an $\mathbf{L}_\beta$–quantum will be very small and the probability of the creation of an $\mathbf{L}_\alpha$–quantum will be practically equal to unity.

In this manner in both cases (a) and (b) the final product is an $\mathbf{L}_\alpha$–quantum. It is easily seen that our considerations generalize to the cases where the transitions are accompanied, instead of the emission of an $\mathbf{L}_\beta$–quantum, by the emission of a $\mathbf{L}_\gamma$, $\mathbf{L}_\delta$, etc. Let p be the probability of Case I and $1 - p$ the probability of Case II.

Thus we see indeed that after the absorption of an ultra–violet quantum there is a finite proba-

bility p of re–emission of it with the same wave–length[1] and a finite probability $1-p$ of re–emission of a quantum $\mathbf{L}_\alpha$. We shall not take into account the intermediate stages in which the absorbed quantum may appear as $\mathbf{L}_\beta$–, $\mathbf{L}_\gamma$–quantum, etc. These will have little influence on the results. The quanta $\mathbf{L}_\alpha$ cannot suffer new transformations and can only be scattered.
The problem in this way is reduced to the study of two superposed fields of radiation: the field of ultra–violet quanta and the $\mathbf{L}_\alpha$–radiation field in the planetary nebulae.

The Field of Ultra–violet Quanta. In this paper we shall use the method of reduction of a spherical problem to a plane problem, developed by Professor Milne. Let k be the absorption coefficient of the ultra–violet radiation per atom. This coefficient depends upon the wave–length. We shall use its mean value. Let, further, n be the number of **H**–atoms in the first level in $1cm^3$, r_1 and r_2 the distances of the inner and outer boundary of the nebular ring from the central star. Then the optical depth at the distance r from the central star is

$$\tau = \int_{r_1}^{r_2} nk\, dr. \tag{1}$$

The equations of transfer of the energy of ultra–violet quanta may be written in Milne form

$$\frac{1}{2}\frac{dI(\tau)}{d\tau} = I(\tau) - B(\tau) \tag{2}$$

$$\frac{1}{2}\frac{dI'(\tau)}{d\tau} = B(\tau) - I'(\tau) \tag{3}$$

if we use an approximation of Schwarzschild–Schuster type.
Here $I(\tau)$ is the average intensity of the diffuse ultra–violet radiation of the nebula in the outward direction at point τ, and $I'(\tau)$ is the average intensity of the same radiation at the same point in the inward direction. The quantity $4\pi\, B(\tau)\, d\tau$ is the amount of energy of ultra–violet quanta emitted in the layer $d\tau$ per second. The same layer absorbs the diffuse ultra–violet radiation from various parts of the nebular ring. The absorbed energy is equal to $2\pi[I(\tau)+I'(\tau)]\, d\tau$.Besides this, the layer absorbs the radiation of the central star. Let πS be the amount of ultra–violet energy falling on each square centimeter of the inner surface of the nebula. At the point τ this amount is reduced to $\pi S\, \exp\left(-(\tau_1-\tau)\right)$, where

$$\tau_1 \equiv \int_{r_1}^{r_2} nk\, dr. \tag{4}$$

is the optical thickness of the nebula. From this amount our layer absorbs

$$\pi S\, \exp\left(-(\tau_1-\tau)\right) d\tau.$$

[1]Owing to the free–free transition as well as to the inelastic collisions of free electron the wave–length of the re–emitted quantum may differ considerably from the wave–length of the absorbed quantum. But it always remains shorter than $\lambda_0 = \frac{c}{\nu_0}$ where c is the velocity of light. In this paper we make no difference between the ultra–violet quanta of different wave–lengths.

Since from the quanta absorbed only the fraction p is re–emitted again as ultra–violet quanta, the equation of radiative equilibrium may be written:

$$p\left[I(\tau)+I'(\tau)+\frac{1}{2}\cdot S\,e^{(-(\tau_1-\tau))}\right]=2\,B(\tau). \tag{5}$$

Introducing the boundary conditions

$$I'(0)=0, \quad I(\tau_1)=I'(\tau_1) \tag{6}$$

we take into account the diffuse radiation incident on any portion of the inner face of the nebular shell and arriving from other portions of the inner face.
From the equations (2) and (3) we have

$$\frac{1}{2}\cdot\frac{d(I+I')}{d\tau}=I-I' \tag{7}$$

$$\frac{1}{2}\cdot\frac{d(I-I')}{d\tau}=I+I'-2B \tag{8}$$

Differentiating (7) and comparing with (6) we obtain

$$\frac{1}{4}\cdot\frac{d^2(I+I')}{d\tau^2}=I+I'-2B \tag{9}$$

Substituting (5) into (9) we find the following equation for $I+I'$:

$$\frac{1}{4}\cdot\frac{d^2(I+I')}{d\tau^2}=(1-p)\,(I+I')-\frac{p}{2}\cdot S\,e^{(-(\tau_1-\tau))}. \tag{10}$$

The general solution of this equation is

$$I+I'=Aexp(-\lambda\tau)+B\exp(\lambda\tau)+\frac{2p}{3-4p}\,Se^{(-(\tau_1-\tau))}, \tag{11}$$

where A and B are constants of integration and $\lambda=2\sqrt{1-p}$. Introducing (11) in (5) we find

$$B(\tau)=\frac{p}{a}\left(A\,exp(-\lambda\tau)+B\exp(\lambda\tau)+\frac{3}{2(3-4p)}\cdot Se^{(-(\tau_1-\tau))}\right). \tag{12}$$

Introducing (11) in (7) we obtain

$$I(\tau)-I'(\tau)=-\frac{\lambda}{2}A\,exp(-\lambda\tau)+\frac{\lambda}{2}B\exp(\lambda\tau)+\frac{p}{3-4p}\cdot Se^{(-(\tau_1-\tau))}. \tag{13}$$

Adding and subtracting (11) and (13) we find $I(\tau)$ and $I'(\tau)$:

$$I(\tau)=\frac{1}{2}\left(1-\frac{\lambda}{2}\right)A\,e^{-\lambda\tau}+\frac{1}{2}\left(1+\frac{\lambda}{2}\right)B\,e^{(\lambda\tau)}+\frac{3p}{2(3-4p)}\cdot Se^{(-(\tau_1-\tau))}, \tag{14}$$

$$I'(\tau) = \frac{1}{2}\left(1+\frac{\lambda}{2}\right) A\, e^{-\lambda\tau} + \frac{1}{2}\left(1-\frac{\lambda}{2}\right) B\, e^{(\lambda\tau)} + \frac{p}{2(3-4p)} \cdot Se^{(-(\tau_1-\tau))}, \tag{15}$$

The first of the conditions (6) may be written according to (15) in the form:

$$A\left(1+\frac{\lambda}{2}\right) + B\left(1-\frac{\lambda}{2}\right) + \frac{3}{3-4p} \cdot S\, e^{-\tau_1} = 0. \tag{16}$$

The second of the conditions (6) gives

$$\lambda B \exp(\lambda\tau_1) + \frac{2p}{3-4p} = \lambda A \cdot exp(-\lambda\tau). \tag{17}$$

From the equations (16) and (17) we find the following coefficients A and B.

$$A = \frac{\left(1-\frac{\lambda}{2}\right) e^{\tau_1} - \frac{\lambda}{2} e^{\lambda\tau_1}}{\frac{\lambda}{2}\left[\left(1-\frac{\lambda}{2}\right)\exp(-\lambda\tau_1) + \left(1+\frac{\lambda}{2}\right)\exp(\lambda\tau_1)\right]} \cdot \frac{p}{3-4p} \cdot Se^{-\tau}, \tag{18}$$

$$B = -\frac{\left(1+\frac{\lambda}{2}\right) e^{\tau_1} + \frac{\lambda}{2} \exp(-\lambda\tau_1)}{\frac{\lambda}{2}\left[\left(1-\frac{\lambda}{2}\right)\exp(-\lambda\tau_1) + \left(1+\frac{\lambda}{2}\right)\exp(\lambda\tau_1)\right]} \cdot \frac{p}{3-4p} \cdot Se^{-\tau_1}, \tag{19}$$

These values of A and B introduced in (12), (14) and (15) give the solution for the field of the ultra–violet quanta.

For the net flux of the diffuse ultra–violet radiation at the outer boundary of the nebula we obtain:

$$\begin{aligned} \pi\, F_u &= \pi\,[I(0) - I'(0)] = \\ &\pi\left[1 - \frac{2e^{\tau_1} - \lambda\sinh(\lambda\tau_1)}{\left(1-\frac{\lambda}{2}\right) exp(-\lambda\tau) + \left(1+\frac{\lambda}{2}\right)\exp(\lambda\tau_1)}\right] \cdot \frac{p}{3-4p} \cdot Se^{-\tau_1}. \end{aligned} \tag{20}$$

For the representation of the solution in numerical form it is necessary to know p and τ_1. The value of p can be calculated from pure physics. Cillié [3] has computed the relative probabilities of the capture of electrons by protons on different hydrogen levels. From his results we have deduced the fraction of captured electrons which pass immediately from a free state to the first level, re–emitting the ultra–violet quanta. This fraction is our p. The value of p depends on the temperature of free electrons. For different temperatures we have:

T	$10,000^\circ$	$20,000^\circ$	$50,000^\circ$
p	0.46	0.49	0.57

Putting $p = 0.5$ we find for large values of τ_1 ($\tau_1 > 3$) the following assymptotic expression for the net flux $\pi\, F_u$ at the outer boundary:

$$\pi\, F_u = 0.7\,\pi\, S\, e^{-\tau_1}.$$

The net flux of the direct ultra–violet radiation of a star will be simply $\pi\, S\, e^{-\tau_1}$ while the whole net flux will be $1.7\,\pi\, S\, e^{-\tau_1}$.

In the absence of the absorbing shell the net flux from the star is πS. The fraction $1-1.7\,e^{-\tau_1}$ of it is converted into other forms of radiation. Owing to the fact that by the splitting of an ultra–violet quantum an $\mathbf{L}_\alpha$–quantum is certainly created, the net flux in the line $\mathbf{L}_\alpha$ at the outer boundary will contain $\frac{1-1.7\,e^{-\tau_1}}{h\,\nu_c}\cdot\mathbf{L}_\alpha$–quanta, where ν_c is the average frequency of the ultra–violet quanta. If τ_1 is large, the flux of the $\mathbf{L}_\alpha$ energy at the outer boundary of nebula is approximately $\frac{\nu_\alpha}{\nu_c}\,\pi S$. Here ν_α is the frequency of the line $\mathbf{L}_\alpha$.

The Field of $\mathbf{L}_\alpha$–radiation . Let κ be the absorption–coefficient within the line $\mathbf{L}_\alpha$ per hydrogen atom in the normal state. The optical depth for this line is defined by

$$t=\int_r^{r_2} n\kappa\,dr. \tag{21}$$

The ratio $\frac{\kappa}{k}=\omega$ may be assumed constant, when the temperature variations within the nebula are neglected. In fact, κ, is a function of atomic constants and of the Doppler breadth of the line only. This breadth depends upon the temperature. When $\frac{\kappa}{k}=\omega$ is constant, the ratio $\frac{t}{\tau}$ is also constant, and we have

$$\frac{t}{\tau}=\frac{\kappa}{k}=\omega. \tag{22}$$

If the temperature of the nebula is of the order 10^3-10^4 degrees, the quantity ω will also be of the order 10^3-10^4. Since we have supposed that the optical thickness τ_1 of the nebula in the ultra–violet region is of the order of unity or larger, the optical thickness in the line $\mathbf{L}_\alpha$,

$$t_1=\int_{r_1}^{r_2} n\kappa\,dr$$

will be of the order 10^3-10^4, or larger.

The equation of transfer of the radiation in the line $\mathbf{L}_\alpha$ has the same form as (2) and (3). Let $K(t)$ be the average intensity of the diffuse $\mathbf{L}_\alpha$–radiation of nebula in the outward direction at the point t, and $K'(t)$ be the average intensity of the same radiation at the same point in the inward direction. The equations of transfer are:

$$\frac{1}{2}\frac{dK(t)}{dt}=K(t)-C(t), \tag{23}$$

$$\frac{1}{2}\frac{dK'(t)}{dt}=C(t)-K'(t), \tag{24}$$

where $4\pi C(t)\,dt$ is the amount of energy emitted by the layer dt in the line $\mathbf{L}_\alpha$ per second. This layer absorbs the diffuse $\mathbf{L}_\alpha$–radiation from the other parts of nebula. The quantity of diffuse radiation absorbed is $2\pi\,[K(t)+K'(t)]\,dt$. The number of $\mathbf{L}_\alpha$–quanta emitted by the central star is negligible, since the number of ultra–violet quanta transformed into $\mathbf{L}_\alpha$–quanta is some thousand times larger.

The number of ultra–violet quanta which are absorbed in the layer $d\tau$ and are transformed into $\mathbf{L}_\alpha$–quanta is

$$\frac{(1-p)[2\pi(I+I') + \pi\, Se^{(-(\tau_1-\tau))}]\, d\tau}{h\,\nu_c}$$

Thus the $\mathbf{L}_\alpha$–radiation created in dt according to (5) is

$$\frac{1-p}{p} \cdot \frac{\nu_\alpha}{\nu_c} \cdot 4\pi\, B(\tau)\, d\tau = 4\pi\, \frac{1-p}{p} \cdot \frac{\nu_\alpha}{\nu_c} \cdot B(\tau)\, \frac{dt}{\omega},$$

Hence the equation of radiative equilibrium is

$$4\pi\, C(t)\, dt = 2\pi\, [K(t) + K'(t)]\, dt + 4\pi\, \frac{1-p}{p} \cdot \frac{\nu_\alpha}{\nu_c} \cdot B(\tau)\, \frac{dt}{\omega},$$

or

$$C(t) = \frac{1}{2}[K(t) + K'(t)] + \frac{\nu_\alpha}{\nu_c} \cdot \frac{1-p}{p\,\omega}\, B(\tau). \tag{25}$$

The boundary conditions are

$$K'(0) = 0, \quad K'(t_1) = K(t_1). \tag{26}$$

Fron the equations (23) and (24) we have

$$\frac{1}{2}\frac{d(K+K')}{dt} = K - K', \tag{27}$$

$$\frac{1}{2}\frac{d(K-K')}{dt} = K + K' - 2C(t), \tag{28}$$

Differentiating (27), and using (28) we find

$$\frac{1}{4}\frac{d^2(K+K')}{dt^2} = K + K' - 2C(t), \tag{29}$$

or according to (25)

$$\frac{1}{4}\frac{d^2(K+K')}{dt^2} = -\frac{2\nu_\alpha}{\nu_c} \cdot \frac{1-p}{p\,\omega}\, B(\tau). \tag{30}$$

Writing $B(\tau)$ in the form

$$\begin{aligned} B(\tau) &= \frac{p}{2}\left(A\, e^{-\lambda\tau} + Be^{\lambda\tau} + De^{(-(\tau_1-\tau))}\right) = \\ &= \frac{p}{2}\left(A\, e^{-\frac{\lambda}{\omega}\, t} + B\, e^{\frac{\lambda}{\omega}\, t} + D\, e^{-\frac{t_1-t}{\omega}}\right), \end{aligned} \tag{31}$$

where

$$D = \frac{3}{2(3-4p)} \cdot S \tag{32}$$

we find the following solution of equation (30):

$$K(t) + K'(t) = a + bt - \frac{4\nu_\alpha}{\nu_c} \cdot \frac{1-p}{\lambda^2}\, \omega\, \left(A\, e^{-\frac{\lambda}{\omega}\, t} + B\, e^{\frac{\lambda}{\omega}\, t} + D\lambda^2\, e^{-\frac{t_1-t}{\omega}}\right),$$

where a and b are constants of integration.

Differentiating this expression and using (27) we find

$$K(t) - K'(t) = \frac{b}{2} - \frac{2\nu_\alpha}{\nu_c} \cdot \frac{1-p}{\lambda} \left(-A\, e^{-\frac{\lambda}{\omega}\, t} + B\, e^{\frac{\lambda}{\omega}\, t} + D\lambda^2\, e^{-\frac{t_1 - t}{\omega}}\right),$$

According to the definition of λ,

$$\lambda = 2\sqrt{1-p}.$$

Therefore

$$K(t) + K'(t) = a + bt - \frac{\nu_\alpha}{\nu_c}\,\omega \left(A\, e^{-\frac{\lambda}{\omega}\, t} + B\, e^{\frac{\lambda}{\omega}\, t} + D\lambda^2\, e^{-\frac{t_1 - t}{\omega}}\right), \tag{33}$$

$$K(t) - K'(t) = \frac{b}{2} - \frac{\nu_\alpha}{\nu_c} \cdot \frac{\lambda}{2} \left(-A\, e^{-\frac{\lambda}{\omega}\, t} + B\, e^{\frac{\lambda}{\omega}\, t} + D\lambda^2\, e^{-\frac{t_1 - t}{\omega}}\right). \tag{34}$$

From (33) and (34) we have

$$\begin{aligned} K(t) = {} & \frac{a}{2} + \frac{b}{4} + \frac{b}{2}\, t - \\ & - \frac{\nu_\alpha}{2\nu_c}\,\omega \left[A\left(1 - \frac{\lambda}{2\omega}\right) e^{-\frac{\lambda}{\omega}\, t} + B\left(1 + \frac{\lambda}{2\omega}\right) e^{\frac{\lambda}{\omega}\, t} + D\lambda^2 \left(1 + \frac{\lambda}{2\omega}\right) e^{-\frac{t_1 - t}{\omega}}\right], \end{aligned} \tag{35}$$

$$\begin{aligned} K'(t) = {} & \frac{a}{2} - \frac{b}{4} + \frac{b}{2}\, t - \\ & - \frac{\nu_\alpha}{2\nu_c}\,\omega \left[A\left(1 + \frac{\lambda}{2\omega}\right) e^{-\frac{\lambda}{\omega}\, t} + B\left(1 - \frac{\lambda}{2\omega}\right) e^{\frac{\lambda}{\omega}\, t} + D\lambda^2 \left(1 - \frac{\lambda}{2\omega}\right) e^{-\frac{t_1 - t}{\omega}}\right]. \end{aligned} \tag{36}$$

The conditions (26) are reduced to

$$\frac{a}{2} - \frac{b}{4} - \frac{\nu_\alpha}{2\nu_c}\,\omega \left[A\left(1 + \frac{\lambda}{2\omega}\right) + B\left(1 - \frac{\lambda}{2\omega}\right) + 4D(1-p)\left(1 - \frac{\lambda}{2\omega}\right) e^{-\frac{t_1}{\omega}}\right] = 0, \tag{37}$$

and

$$b = \frac{\nu_\alpha}{\nu_c} \left[-\lambda A\, e^{-\lambda\tau_1} + \lambda B\, e^{\lambda\tau_1} + 4D(1-p)\right]. \tag{38}$$

Introducing (38) in (37) we find:

$$\begin{aligned} a = {} & \frac{\nu_\alpha}{2\nu_c} \left[-\lambda A\, e^{-\lambda\tau_1} + \lambda B\, e^{\lambda\tau_1} + 4D(1-p)\right] + \\ & + \frac{\nu_\alpha}{2\nu_c}\,\omega \left[A\left(1 + \frac{\lambda}{2\omega}\right) + B\left(1 - \frac{\lambda}{2\omega}\right) + 4D(1-p)\left(1 - \frac{\lambda}{2\omega}\right) e^{-\frac{t_1}{\omega}}\right]. \end{aligned} \tag{39}$$

The equations (35) and (36) together with (38) and (39) give the solution for the $\mathbf{L}_\alpha$–field.

The Density of Radiation in the Inner Layers of the Nebula. We have denoted above by $\pi\, S$ the amount of energy of the ultra–violet radiation falling from the star on each square centimeter of the inner surface of the nebula. In the absence of re–emission the mean intensity of the ultra–violet radiation in this region will be equal $\frac{\pi\, S}{4\pi} = 0.25\, S$. In the case where re–emission is taken into account, the average intensity of ultra–violet radiation increases and is equal to $\frac{1}{4}\,S+\frac{1}{2}(I_1+I_2)$. According to (11), (18) and (19)

$$I(\tau_1)+I'(\tau_1)=\frac{2p}{3-4p}\cdot S\left[1-\frac{1}{\lambda}\cdot\frac{\lambda\,e^{-\lambda_1}+\lambda\,\cosh(\lambda\tau_1)+2\sinh(\lambda\tau_1)}{2\cosh(\lambda\tau_1)+\lambda\sinh(\lambda\tau_1)}\right]. \tag{40}$$

The expression in brackets remains between 0 and $1-\frac{1}{\lambda}$, when τ_1 changes between 0 and ∞. Therefore we have

$$\frac{1}{4}\cdot S\leq\frac{1}{4}\cdot S+\frac{1}{2}(I_1+I_2)\leq\frac{1}{4}\cdot S+\frac{2p\left(1-\frac{1}{\lambda}\right)}{3-4p}\cdot S.$$

Putting $p=0.5$ we obtain:

$$\frac{1}{4}\cdot S\leq\frac{1}{4}\cdot S+\frac{1}{2}(I_1+I_2)<0.40\,S.$$

Thus the mean intensity of ultra–violet radiation at the inner boundary of nebula is of the same order of magnitude as in the absence of nebular shell. It should be doubled, if τ_1 is very large.

The state of affairs entirely changes, when we consider the $\mathbf{L}_\alpha$–field. Owing to the large optical thickness of the nebula in the $\mathbf{L}_\alpha$–line, and to the fact that all $\mathbf{L}_\alpha$–quanta absorbed are re–emitted in the same frequency, the density of $\mathbf{L}_\alpha$-radiation in the inner layers of the ring is very large.
In order to estimate this density we consider a modification of (38) and (39). In fact, comparing (38) with (17) and (32) we find

$$b=\frac{2\nu_\alpha}{\nu_c}\cdot S \tag{41}$$

and neglecting in (39) the terms not containing the factor ω

$$a=\frac{\nu_\alpha}{\nu_c}\,\omega\cdot\left[A+B+4D(1-p)\,e^{-\tau_1}\right]. \tag{42}$$

Substituting the values of A, B and D we find:

$$a=\frac{\nu_\alpha}{\nu_c}\,\omega\cdot\left[3(1-p)\,e^{-\tau_1}-\frac{1+e^{-\tau_1}\cosh(\lambda\tau_1)}{2\cosh(\lambda\tau_1)+\lambda\sinh(\lambda\tau_1)}\right]\frac{2p}{3-4p}\cdot S. \tag{43}$$

For the brackets in (33) we have

$$\begin{aligned}&A\,e^{-\lambda\tau_1}+B\,e^{\lambda\tau_1}+D\,\lambda^2=\\&\frac{S}{3-4p}\left[6(1-p)-\frac{2p}{\lambda}\cdot\frac{\lambda\,e^{-\lambda_1}+\lambda\,\cosh(\lambda\tau_1)+2\sinh(\lambda\tau_1)}{2\cosh(\lambda\tau_1)+\lambda\sinh(\lambda\tau_1)}\right].\end{aligned} \tag{44}$$

This expression varies within the limits

$$2S\leq A\,e^{-\lambda\tau_1}+B\,e^{\lambda\tau_1}+D\,\lambda^2<\frac{S}{3-4p}\left[6(1-p)-\frac{2p}{\lambda}\right]$$

or, if $p=0.5$

$$2S\leq A\,e^{-\lambda\tau_1}+B\,e^{\lambda\tau_1}+D\,\lambda^2\leq 2.29\,S$$

and in the first approximation

$$A\,e^{-\lambda\tau_1} + B\,e^{\lambda\tau_1} + D\,\lambda^2 = 2.15\,S. \tag{45}$$

For the mean intensity of $\mathbf{L}_\alpha$–radiation at the inner boundary we obtain approximately:

$$\frac{1}{2}\,[K(t) + K'(t)] = \frac{\nu_\alpha}{\nu_c}\,\omega\,S \cdot \left[\tau_1 + \frac{1}{2}f(\tau_1) - 1.07\right],$$

where

$$f(\tau_1) = \left[3(1-p)\cdot e^{-\tau_1} - \frac{1 + e^{-\tau_1}\cosh(\lambda\tau_1)}{2\cosh(\lambda\tau_1) + \lambda\sinh(\lambda\tau_1)}\right].$$

If $\tau \geq 2$ we may neglect $\frac{1}{2}f(\tau_1)$ and have approximately

$$\frac{1}{2}\,[K(t) + K'(t)] = \frac{\nu_\alpha}{\nu_c}\,\omega\,S \cdot [\tau_1 - 1].$$

We may take $\omega = 1000$ (see [4]). Therefore if $\tau_1 = 2$ the mean density of the $\mathbf{L}_\alpha$–radiation at the inner boundary will be of the order $1000\,S$, where πS is again the energy of the whole ultra–violet radiation falling on each square centimeter of the inner surface of the nebula from the central star. Therefore the density of $\mathbf{L}_\alpha$–radiation in this example is 10000 times larger, than the density of the whole diluted ultra–violet radiation of the nucleus in the absence of the absorbing shell at the same distance. A rough estimate shows that the ultra–violet radiation of the black body at the temperatures of the order $40000^\circ - 50000^\circ$ is about $5 \cdot 10^4$ times stronger than the same radiation within the Doppler width of the $\mathbf{L}_\alpha$–line corresponding to the temperature of the nebular matter. Thus the density of $\mathbf{L}_\alpha$–radiation in the inner layers of the nebular envelope will be

$$10000 \times 5 \cdot 10^4 = 5 \cdot 10^8$$

times larger than the density of the direct $\mathbf{L}_\alpha$–radiation of the central star within the Doppler line–width.

Such a large density of $\mathbf{L}_\alpha$–radiation will produce a large accumulation of atoms in the state $2P$. On the other hand there will be also a large accumulation of hydrogen atoms in the metastable state $2S$. It may occur therefore that the optical thickness of the nebula in the lines of the Balmer series will not be very small.

Radiation Pressure in the Outer Parts of the Nebula. The greater part of the ultra–violet radiation of the star is transformed by the nebula into $\mathbf{L}_\alpha$–quanta. Therefore the flux of radiation emitted by the nebula will consist chiefly of $\mathbf{L}_\alpha$–quanta. For sufficiently large τ_0 each ultra–violet quantum will give rise to an $\mathbf{L}_\alpha$–quantum, and the flow of $\mathbf{L}_\alpha$–radiation from the nebula will be of the order $\frac{\nu_\alpha}{\nu_c} \cdot \pi\,S$. On the inner surface of the nebula the flux of $\mathbf{L}_\alpha$–radiation will be practicaly equal to zero, and the flux of ultra–violet radiation will be πS. Now the radiation pressure in a layer of gas is proportional to the absorption coefficient. On the inner surface of the nebula the resulting flux of radiation consists of ultra–violet quanta, for which the absorption coefficient is small. Therefore the radiation pressure will not be very large. In the outer parts of the nebula, on the contrary, the flux of radiation consists chiefly of $\mathbf{L}_\alpha$–quanta, and the absorption coefficient is about 10^4 times larger than in the case of ultra–violet quanta, while the flux of radiation is of the same order. The radiation pressure, or more exactly, the gradient of radiation pressure, will be here 10^4 times larger than on the inner boundary of the ring. It is physically clear that for τ_0 large the net flux of $\mathbf{L}_\alpha$–radiation πF_α in the outer layer of the ring will be determined by

$$\pi F_\alpha = \left(\frac{r}{r_n}\right)^2 \cdot \frac{2\pi h\nu_\alpha}{c^2} \int_{\nu_0}^{\infty} \frac{\nu^2\, d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1},$$

where r and r_n are respectively the radius of the central star and the radius of the nebula. The average impulse received by a hydrogen atom in a normal state per second from $\mathbf{L}_\alpha$–quanta will be

$$\frac{\kappa\pi F_\alpha}{c} = \left(\frac{r}{r_n}\right)^2 \cdot \frac{2\kappa\pi h\nu_\alpha}{c^3} \int_{\nu_0}^{\infty} \frac{\nu^2\, d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}.$$

The impulse received by each hydrogen atom from the gravitational field of the central star per second is

$$g\left(\frac{r}{r_n}\right)^2 m,$$

where g the gravitational acceleration on the surface of the central star. However not only the normal hydrogen atoms, but also the free protons are subject to gravitational force. Therefore the ratio μ of repulsive force R to the attractive force G is given by

$$\mu = \frac{R}{G} = \frac{\kappa\,\pi}{mg\left(1 + \frac{n^+}{n_1}\right)} \frac{2h\nu_\alpha}{c^3} \int_{\nu_0}^{\infty} \frac{\nu^2\, d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1},$$

where n^+ is the number of protons per cubic centimeter, n^- is the number of normal hydrogen atoms in the same volume. Even in the case when we put

$$\frac{n^+}{n_1} = 500$$

which value is probably too high (see [5]), we obtain for $T = 40000^\circ$.

$$\mu = \frac{10^{10}}{g}.$$

The value of g for the nuclei of planetary nebulae will be much larger than that for the Sun. But it is very improbable that it may reach 10^{10} cm. sec^{-2}. Therefore we may conclude that if the optical thickness of the nebula in the ultra–violet region is not too small, the radiation pressure will be the dominant factor in the exterior parts of the nebula.

Remark. The above is a reproduction of part I of a longer paper under the same title published in Pulkovo Bulletin in 1933. The first version of this part was published in Monthly Notices, [1].

The essential point was the study of $\mathbf{L}_\alpha$–field. The second part of the Pulkovo paper was devoted to helium nebulae.

Here it is omitted since after the appearence of our paper 60 years ago more complicated problems were discussed, and solved by other authors (the most important steps have been done by V. V. Sobolev).

R E F E R E N C E S

1. V. A. Ambartzumian, M. N., 93, 50, 1932.
2. Milne, Z. f. Astroph., 1, 105, 1980.
3. Cillié, M. N., 92, 820, 1932.
4. M.N. 93, 58, 1932.
5. Zanstra, Z. f. Astroph., 2, 337, 1931.

Astronomical Observatory
of Leningrad University.