ON THE SCATTERING OF LIGHT BY PLANETARY ATMOSPHERES

The problem of scattering and absorbtion of light by planetary atmospheres has been the subject of many theoretical studies. However, because of mathematical difficulties, no satisfactory solution adequate to the real physical conditions has been found.

It is known that the problem of distribution of brightness on the planetary disc in some approximation is equivalent to the problem of diffuse reflection from a turbid plane–parallel layer of definite optical thickness. In its turn, the problem of diffuse reflection from a turbid medium requires to take into account the multiple elementary processes of scattering.

In an earlier paper [1] the author has given an exact solution for the case where the medium stretches to infinity in both mutually opposite directions and for any scattering indicatrix. At the same time it became a basis for finding a solution of diffuse reflection problem for a layer which streches to infinity only in one direction by means of successive approximations.

In the present paper a new and more exact solution is given to the problem by reducing it to an easily numerically solvable functional equation. One of the advantages of this method is that the unknown function can be found without intermediate calculation of different functions describing the radiation field in the inner layers of the medium.

The method is not restricted to the case of spherical indicatrix of scattering. However in order to simplify the presentation, we treat here the case of the spherical indicatrix only, postponing the treatment of general case to another occasion. We show that at the same time, the method yields a solution to the problem of distribution of brightness over the solar disc.

§1. Integral Equation of the Problem of the Diffuse Reflection

Let us consider plane–parallel layers of matter which is able both to scatter and to absorb passing radiation. Let this matter fill a halfspace (one sided infinity). Assume that the ratio of the absorbtion coefficient to the extinction coefficient (the latter is a sum of absorbtion and extinction coefficients) is a constant to be denoted by λ. We suppose that on the plane bounding the medium falls radiation under angle θ_0 to the normal.Let the density of radiation flow (the energy passing through unit area perpendicular to the flow) be πS.

The integral equation of the theory of scattering in the case of spherical indicatrix is known to have the form

$$B(\tau) = \frac{\lambda}{4}\, S \exp\left(-\tau \sec\theta_0\right) + \frac{\lambda}{2} \int_0^{\infty} \mathrm{Ei}\,|\tau - t|\, B(t)\, dt, \tag{1}$$

where $B(\tau) = \eta / \alpha$ is the ratio coefficient of radiation / extinction coefficient and τ is the optical depth. If the solution of (1) is known, the intensity of light diffusely reflected in the direction making an angle θ with the normal can be found from the formula

$$I(\theta_1) = \int_0^\infty \exp(-\tau \sec\theta_0)\, B(\tau) \sec\theta_1 \, d\tau. \tag{2}$$

The usual method consists in finding a solution of (1) and substituting $B(\tau)$ into (2). In this way one can find the intensity I as a function of the angles of incidence θ_0 and reflection θ_1. From the linearity of the problem it is clear that both $B(\tau)$ and I will be proportional to S. Let us denote

$$\frac{I}{S} = r(\theta_1, \theta_0).$$

This quantity will depend on θ_1 and θ_0 but not on S.

The problem of diffuse reflection requires to find the function $r(\theta_1, \theta_0)$. Bellow we derive an equation directly for $r(\theta_1, \theta_0)$ and find its solution.

§2. The functional equation, determining the function $r(\theta_1, \theta_0)$

Let us set in (1) $\xi = \sec\theta_0$ and $\tau = \sigma + a,\ t = s + a$:

$$B(\sigma + a) = \frac{\lambda}{4} \exp(-\xi(\sigma + a)) + \frac{\lambda}{2} \int_{-a}^\infty \mathrm{Ei}\,|\sigma - s|\, B(s + a)\, ds, \tag{3}$$

and take for a moment $S = 1$. Differentiating this equation by a we have

$$B'(\sigma + a, \xi) - \frac{\lambda}{2} \int_{-a}^\infty \mathrm{Ei}\,|\sigma - s|\, B'(s + a, \xi)\, ds =$$
$$= -\frac{\lambda}{4} \exp(-\xi(\sigma + a)) + \frac{\lambda}{2} \mathrm{Ei}(\sigma + a)\, B(0, \xi),$$

where we show explicitly the dependence of B from the parameter $\xi = \sec\theta_0$. Putting $a = 0$ we obtain

$$B'(\sigma, \xi) - \frac{\lambda}{2} \int_0^\infty \mathrm{Ei}\,|\sigma - s|\, B'(s, \xi)\, ds = -\frac{\lambda}{4} \exp(-\xi\sigma) + \frac{\lambda}{2} \mathrm{Ei}(\sigma)\, B(0, \xi). \tag{4}$$

But

$$\mathrm{Ei}(\sigma) = \int_1^\infty \exp(-\sigma\xi)\, \frac{d\zeta}{\zeta},$$

and we conclude that the right hand side of (4) is a superposition of terms of the type $\exp(-\sigma\xi)$. The same is true for the right hand side of (1). Owing to the linearity, we can write the solution of (4) as a superposition of solutions of the equations of type (1):

$$B'(\sigma, \xi) = -\xi B(\sigma, \xi) + 2B(0, \xi) \int_1^\infty B(\sigma, \zeta)\, \frac{d\zeta}{\zeta}. \tag{5}$$

Multiplying (5) by $\exp(-\eta\sigma)$ and integrating over σ, we find

$$\begin{aligned}\int_0^\infty \exp(-\eta\sigma) B'(\sigma,\xi)\,d\sigma = -\xi \int_0^\infty \exp(-\eta\sigma)\, B(\sigma,\xi)\,d\sigma + \\ + 2B(0,\xi)\int_1^\infty \frac{d\zeta}{\zeta}\int_0^\infty B(\sigma,\zeta)\,\exp(-\eta\sigma)\,d\sigma.\end{aligned} \tag{6}$$

Integrating in the left hand side we can exclude the derivative of B

$$\begin{aligned}\eta\int_0^\infty \exp(-\eta\sigma)\, B(\sigma,\xi)\,d\sigma - B(0,\xi) = -\xi \int_0^\infty \exp(-\eta\sigma) B(\sigma,\xi)\,d\sigma + \\ + 2B(0,\xi)\int_1^\infty \frac{d\zeta}{\zeta}\int_0^\infty B(\sigma,\zeta)\,\exp(-\eta\sigma)\,d\sigma.\end{aligned} \tag{7}$$

However

$$\eta\int_0^\infty \exp(-\eta\sigma)\, B(\sigma,\xi)\,d\sigma = I(\eta) = r(\eta,\xi),$$

since we have adopted $S = 1$.

Therefore the equation (7) can be rewritten in the form

$$\frac{\xi+\eta}{\eta} r(\eta,\xi) = B(0,\xi)\left[1 + \frac{2}{\eta}\int_1^\infty r(\eta,\zeta)\,\frac{d\zeta}{\zeta}\right].$$

For the function

$$R(\eta,\xi) = \frac{r(\eta,\xi)}{\eta}$$

we now find

$$(\xi+\eta)R(\eta,\xi) = B(0,\xi)\left[1 + 2\int_1^\infty R(\eta,\xi)\,\frac{d\zeta}{\zeta}\right]. \tag{8}$$

On the other hand from (1) we have, for $\tau = 0$

$$B(0,\xi) = \frac{\lambda}{4}\left[1 + 2\int_0^\infty \mathrm{Ei}(t)\, B(t,\xi)\,dt\right],$$

or using the integral expression of $\mathrm{Ei}(t)$

$$B(0,\xi) = \frac{\lambda}{4}\left[1 + \frac{2}{\xi}\int_1^\infty r(\xi,\zeta)\,\frac{d\zeta}{\zeta}\right]. \tag{9}$$

Taking into account that $r(\xi,\zeta) = \xi R(\xi,\zeta)$ we obtain

$$B(0,\xi) = \frac{\lambda}{4}\left[1 + 2\int_1^\infty R(\xi,\zeta)\,\frac{d\zeta}{\zeta}\right]. \tag{9a}$$

Substituting (9a) in (8) we find

$$(\xi+\eta)R(\eta,\xi) = \frac{4}{\lambda}\, B(0,\xi)\; B(0,\eta),$$

$$r(\eta, \xi) = \frac{4}{\lambda} \frac{\eta}{\eta + \xi} B(0, \xi) \ B(0, \eta). \tag{10}$$

Thus we conclude that in the case of spherical indicatrix of scattering the function $r(\eta, \xi)$ of diffuse reflection is represented by product of two identical functions each of which depends on one variable, multiplied by the ratio $\frac{\eta}{\eta + \xi}$. We recall in this connection that Minnaert [2] has recently noted that the ratio $\frac{r(\eta, \xi)}{\eta}$ is always a symmetrical function of η and ξ independently of the form of scattering indicatrix.

In a paper of V. A. Fock (now in press [3]) where the exact solution of (1) was received, it was shown that the ratio $\frac{\eta + \xi}{\eta} r(\eta, \xi)$ is a product of some function of η and of the same function of ξ, and this function was expressed as an integral depending on a parameter. Our aim now is to find a functional equation for $B(0, \xi)$.

SUbstituting (10) into right hand side of (9a) we find

$$B(0, \xi) = \frac{\lambda}{4} \left[1 + \frac{8}{\lambda} B(0, \xi) \int_1^\infty \frac{B(0, \zeta)}{\xi + \zeta} \frac{d\zeta}{\zeta} \right].$$

Instead of $\xi = \sec \theta_0$ we consider $x = 1 / \xi = \cos \theta_0$ to be an argument and denote

$$\frac{2}{\sqrt{\lambda}} B(0, \xi) = \frac{2}{\sqrt{\lambda}} B \left(0, \frac{1}{x} \right) = \varphi(x). \tag{11}$$

Then we find a functional equation for $\varphi(x)$

$$\varphi(x) = \frac{\sqrt{\lambda}}{2} \left[1 + 2\varphi(x) \, x \int_0^1 \frac{\varphi(x)}{x + z} \, dz \right]. \tag{12}$$

Now we can represent $r(\eta, \xi)$ as a function of y and x

$$r(y, x) = \frac{x}{x + y} \varphi(x) \, \varphi(y). \tag{13}$$

Thus the solution of the functional equation (12) will give us immediately the function $r(y, x)$ of diffuse reflection. In the next paragraph we present this solution for different values of λ.

The advantage of this approach is that in this way we avoid consideration of the functions which describe the radiation field inside the medium.

Of course the proposed method of reduction of the integral equation (1) to a functional equation by means of Laplace transformation can be extended to other integral equations which have kernels depending only on difference $\tau - t$.

§3. Solution of the functional equation for $\varphi(x)$

Instead of $\varphi(x)$ we consider a function

$$\psi(x) = \frac{2}{\sqrt{\lambda}} \varphi(x) \tag{14}$$

which obviously satisfies the equation

$$\psi(x) = 1 + \frac{\lambda}{2}\, x\psi(x) \int_0^1 \frac{\psi(z)}{x+z}\, dz. \tag{15}$$

For $\lambda < 1$ the numerical solution of this equation can be by successive approximation. We begin by taking in the right side of (15) the approximation of the zero order $\psi_0(x) = 1$. As the first approximation we obtain

$$\psi_1(x) = 1 + \frac{\lambda}{2}\, x \ln \frac{1+x}{x}$$

The values of the function $\varphi(x) = \frac{\sqrt{\lambda}}{2}\psi(x)$ for different λ are given in the following table.

$\lambda\backslash x$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.519	0.360	0.387	0.403	0.415	0.424	0.431	0.438	0.443	0.448	0.452	0.455
0.612	0.391	0.428	0.449	0.466	0.479	0.490	0.500	0.508	0.516	0.522	0.528
0.728	0.426	0.477	0.508	0.532	0.552	0.570	0.585	0.598	0.609	0.620	0.629
0.806	0.449	0.512	0.553	0.586	0.613	0.637	0.658	0.677	0.694	0.708	0.722
0.865	0.465	0.540	0.590	0.631	0.666	0.697	0.724	0.749	0.771	0.790	0.809
0.910	0.477	0.563	0.621	0.672	0.715	0.754	0.789	0.821	0.851	0.878	0.903
0.944	0.486	0.580	0.647	0.706	0.759	0.805	0.850	0.889	0.926	0.962	0.993
0.969	0.492	0.594	0.669	0.736	0.798	0.852	0.906	0.955	1.001	1.045	1.086
0.986	0.497	0.607	0.691	0.767	0.837	0.903	0.967	1.026	1.081	1.137	1.188
0.997	0.499	0.617	0.710	0.797	0.878	0.958	1.083	1.108	1.179	1.250	1.319
1.000	0.500	0.624	0.725	0.821	0.915	1.007	1.097	1.188	1.276	1.365	1.455

Table 1. Value of $\varphi(x) = \dfrac{\sqrt{\lambda}}{2}\psi(x)$ for different λ.

All our numerical integrations have been carried out using Simpson formula.
Actually the process of successive approximation will converge much more rapidly if we begin not with $\psi_0(x) = 1$ but rather with a function which is more or less near to the exact solution.

For example from (15) it is clear that $\psi(0) = 1$. This suggests taking for $\psi_0(x)$ a linear function

$$\psi_0(x) = 1 + ax,$$

where the constant a can be determined from the condition that the integral of $\psi_0(x)$ over the whole interval is equal to the same integral of the exact solution $\psi(x)$, i.e.

$$1 + \frac{a}{2} = \int_0^1 \psi(x)\, dx. \tag{16}$$

But we are able to find the exact value of integral on the right hand side of (16) in the following way. Let us integrate both sides of (15)

$$\int_0^1 \psi(x)\, dx = 1 + \frac{\lambda}{2} \int_0^1 \int_0^1 \frac{\psi(x)\psi(z)\, z\, dz\, dx}{x+z}.$$

Taking into account that the integral on the right hand side is equal

$$\frac{1}{2} \int_0^1 \int_0^1 \psi(x)\psi(z)\, dz\, dx = \frac{1}{2} \left[\int_0^1 \psi(x)\, dx \right]^2,$$

we obtain a quadratic equation which yields

$$\int_0^1 \psi(x)\, dx = \frac{2}{\lambda} \left(1 - \sqrt{1-\lambda} \right). \tag{17}$$

With the same aim of improving the initial approximation we can use the processes of interpolation and extrapolation as soon as approximate solutions for two different values of λ are calculated. With moderate effort it is possible to find such a $\psi_0(z)$ that ψ_1 differs from ψ_0 not more than in two or three units of the third decimal.

In the case $\lambda > 0.95$ it is better to modify the process in the following way. Under the sign of integral in (15) let us substitute

$$\frac{x}{x+z} = 1 - \frac{z}{x+z}.$$

Then

$$\psi(x) = 1 + \frac{\lambda}{2}\, \psi(x) \int_0^1 \psi(z)\, dz - \frac{\lambda}{2}\, \psi(x) \int_0^1 \frac{\psi(z)\, z\, dz}{x+z},$$

or using (17)

$$\sqrt{1-\lambda}\,\psi(x) = 1 - \frac{\lambda}{2}\psi(x)\int_0^1 \frac{\psi(z)\,z\,dz}{x+z},$$

from which

$$\psi(x) = \left[\sqrt{1-\lambda} + \frac{\lambda}{2}\int_0^1 \frac{\psi(z)\,z\,dz}{x+z}\right]^{-1}. \tag{18}$$

Now putting in the right hand side $\psi(z) = \psi_0(z)$ we calculate $\psi_1(z)$. Then in the right hand side of (18) we put $\psi(z) = \frac{1}{2}\left[\psi_0(z) + \psi_1(z)\right]$ and obtain some function $\psi_2(z)$. Then we again form the mean etc.. When $\lambda > 0.95$ the process is rapidly converging.

Let us note that at $\lambda = 1$ the equation (18) turns into

$$\psi(x)\int_0^1 \frac{\psi(z)\,z\,dz}{x+z} = 2,$$

or

$$\frac{2}{x}\int_0^1 r(z,x)\,z\,dz = 1.$$

Multiplying both sides by $\pi\,Sx$ we find that this equation represents the condition of equality of incident and reflected flows, i.e. that for $\lambda = 1$ the albedo for every angle of incidence equals to 1.

§4. The Distribution of Brightness over the planetary disk

The results obtained enable to determine the distribution of brightness over the disk in different phases. The most simple result we have for the case where the planet is in opposition. In such case $\theta_1 = \theta_0$ and $y = x$. Therefore we obtain

$$r(y,x) = \frac{1}{2}\left[\varphi(x)\right]^2, \tag{13'}$$

where x is the cosin of angular distance from the centre of the disk.

$\lambda\backslash x$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.519	0.065	0.075	0.081	0.086	0.090	0.093	0.096	0.098	0.100	0.102	0.104
0.612	0.076	0.092	0.101	0.108	0.114	0.120	0.125	0.129	0.133	0.137	0.140
0.728	0.091	0.112	0.129	0.141	0.152	0.162	0.171	0.179	0.186	0.192	0.198
0.806	0.101	0.131	0.153	0.172	0.188	0.203	0.216	0.229	0.241	0.251	0.261
0.865	0.108	0.146	0.174	0.199	0.222	0.243	0.262	0.280	0.297	0.312	0.327
0.910	0.114	0.158	0.193	0.226	0.256	0.284	0.311	0.337	0.362	0.385	0.408
0.944	0.118	0.168	0.209	0.249	0.288	0.324	0.361	0.395	0.429	0.463	0.493
0.969	0.121	0.176	0.224	0.271	0.318	0.363	0.410	0.456	0.501	0.546	0.590
0.986	0.124	0.184	0.239	0.294	0.350	0.408	0.468	0.526	0.584	0.646	0.706
0.997	0.125	0.190	0.252	0.318	0.385	0.459	0.534	0.614	0.695	0.781	0.870
1.000	0.125	0.195	0.263	0.337	0.419	0.507	0.602	0.705	0.814	0.931	1.058

Table 2. The theoretical distribution of brightness over the planetary disk in the case of opposition of the planet for different λ, according to (13′).

For the intensity we have

$$I = \frac{1}{2}\left[\varphi(x)\right]^2 S. \tag{19}$$

Consider an absolutely white surface which scatters light according to Lambert law, situated at the distance of the planet from the Sun, perpendicular to solar radiation (we use the language of visual photometry). It is evident that $\frac{1}{2}\left[\varphi(x)\right]^2$ is the ratio of brightness at the point x of the planetary disk to the brightness S of such white surface.

The maximal contrast i.e. the maximal ratio of brightness at the centre to that at the edge of the disk, we have at $\lambda = 1$ (the case of pure scattering). When λ tends to zero the planetary disk is becoming homogeneously bright.

Comparing our results with observations it is important to keep in mind that they are applicable only to gaseous envelopes of great optical thickness such as the atmospheres of Jupiter, Saturn and Venus. One should also not forget that we have supposed that scattering indicatrix is spherical. At the same time we supposed that in all regions of atmosphere the optical properties are identical i.e. we neglected possible presence of local details.

It is quite possible and even probable that the scattering indicatrix is not spherical. The theoretical calculations for non spherical indicatrixes we will give in another paper. With all these reservations, we still have made comparison with observations in the case of Jupiter, using the absolute measurements of brightness published by V. V. Sharonov [4].

Of the greatest importance are comparisons with the absolute values, since otherwise it is always possible to find some λ for which the theoretical ratio centre / edge has the observed value, if only this ratio is between 1.0 and 8.0. Comparison with absolute measurements has the advantage, that from the observed brightness at the centre of the disk ($x = 1.0$) one can determine λ and then find the contrast. This is a more severe test for the theory. For this reason we have taken the observations of Sharonov. Since for the centre of the disk the observations give $r = 0.590$, we concluded that for the atmosphere of Jupiter is $\lambda = 0.969$. It was found that by $\lambda = 0.969$ the theoretical curve $\frac{1}{2}\varphi^2$ represents sufficiently well the distribution of brightness along the equatorial diameter. Only on the edge the discrepancy is larger than 11%. Since the precision of measurements drops at the edges, we can consider the accordance as sufficiently good.

§5. The theoretical albedo

The auxiliary functions we have introduced allow to determine the theoretical value of the albedo i.e. the ratio of the flux reflected by the atmosphere to the incident flux. Generally this theoretical albedo depends on the angle between the incident flux and the direction normal to the layer.

For the flux scattered from the unit surface of the scattering layer we have

$$H = \int I \cos\theta_1 \, d\omega_1 = S \int r(\theta_1, \theta_0) \cos\theta_1 \, d\omega_1,$$

where $d\omega_1$ is an element of solid angle. For the flux of the incident radiation we have

$$F = \pi \, S \cos\theta_0$$

For the albedo we find:

$$A = \frac{H}{F} = \frac{2}{\cos\theta_0} \int r(\theta_1, \theta_0) \cos\theta_1 \sin\theta_1 \, d\theta_1,$$

and since

$$r(\theta_1, \theta_0) = R(\theta_1, \theta_0) \sec\theta_1,$$

we conclude that

$$A = \frac{2}{\cos\theta_0} \int R(\theta_1, \theta_0) \sin\theta_1 \, d\theta_1,$$

or

$$A = \frac{2}{y} \int_0^1 \frac{xy\varphi(x)\varphi(y)}{x+y} \, dx,$$

or

$$A = 2\varphi(y)\int_0^1 \varphi(x)\,dx - 2\varphi(y)\int_0^1 \frac{y\varphi(x)\,dx}{x+y}.$$

By means of (12), (14) and (17) we transform this equation to

$$A = 1 - 2\sqrt{\frac{1}{\lambda} - 1}\,\varphi(y). \tag{20}$$

This is the final expression for albedo. The Table 3 contains the values computed according to (20) for A dependent on y and λ. We see that for the skew rays the albedo is smaller than in the case where the incident flow is normal. This difference is stronger in the case of smaller λ, i.e. for the media with smaller albedo.

$\lambda\backslash x$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.519	0.307	0.255	0.224	0.201	0.183	0.170	0.156	0.146	0.137	0.130	0.124
0.612	0.376	0.318	0.274	0.257	0.236	0.218	0.201	0.190	0.177	0.168	0.158
0.728	0.478	0.414	0.378	0.348	0.324	0.302	0.284	0.257	0.254	0.240	0.230
0.806	0.560	0.498	0.458	0.426	0.399	0.376	0.355	0.336	0.320	0.306	0.292
0.865	0.633	0.573	0.535	0.502	0.474	0.450	0.428	0.408	0.391	0.376	0.361
0.910	0.700	0.646	0.610	0.573	0.551	0.526	0.505	0.484	0.466	0.448	0.433
0.944	0.763	0.717	0.685	0.656	0.680	0.608	0.586	0.567	0.549	0.531	0.516
0.969	0.825	0.788	0.762	0.738	0.716	0.697	0.677	0.660	0.644	0.628	0.614
0.986	0.883	0.857	0.837	0.819	0.802	0.787	0.772	0.758	0.745	0.732	0.720
0.997	0.949	0.936	0.926	0.917	0.909	0.900	0.893	0.885	0.877	0.870	0.863
1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 3. Values of $A = 1 - 2\sqrt{\frac{1}{\lambda} - 1}\,\varphi(y)$ for different λ.

§6. The distribution of the brightness over the solar disk and similar problems

Let us again consider a plane–parallel absorbing and scattering layer. Suppose that the layer has a finite optical thickness and that the sources of radiation lie behind the layer. In this situation we have some flux of radiation which has penetrated through the layer after scatterings. Let us increase the optical thickness of the layer, maintaning everywhere its optical parameters, including the constant ratio of the scattering coefficient to the absorbtion coefficient . At the same time let us increase the radiation power of illuminating sources behind the layer in such proportion that the flux of penetrated radiation remains constant. In the limit we will have a half-space medium and sources situated infinitely deep. Posing in this way the problem of radiation field, we come to the famous integral equation

$$\overline{B}(\tau) = \frac{\lambda}{2} \int_0^\infty \mathrm{Ei}|\tau - t|\, \overline{B}(t)\, dt. \tag{21}$$

In particular, for $\lambda = 1$ we obtain an infinite purely scattering layer and the mathematical problem of E. Milne. Thus our problem for $\lambda = 1$ is equivalent to the problem of radiactive equilibrium of the photosphere, though the physical picture can be quite different. Therfore the angular dependence obtained from (21) yields simultaneously the distribution of brightness over the solar disk.

The intensity of the photospheric radiation in a given direction is determined by the equation

$$r(\eta) = \int_0^\infty \exp(-\eta t)\, \eta \overline{B}(t)\, dt, \tag{22}$$

where η is the secans of the angle between the direction of radiation and the normal to layers. Now $r(\eta)$ may be expressed through the function φ, which was introduced above.

For the derivative $\overline{B}'(\tau)$ we obtain from (21):

$$\overline{B}'(\tau) = \frac{\lambda}{2} \int_0^\infty \mathrm{Ei}|\tau - t|\, \overline{B}'(t)\, dt = \frac{\lambda}{2} \mathrm{Ei}(\tau) \overline{B}(0). \tag{23}$$

Since

$$\mathrm{Ei}(\tau) = \int_1^\infty \exp(-\tau\zeta)\, \frac{d\zeta}{\zeta},$$

we can obtain the solution of (23) as a superposition of solutions of the equations of type (23). This leads to

$$\overline{B}'(\tau) = 2\overline{B}(0) \int_1^\infty B(\tau, \zeta)\, \frac{d\zeta}{\zeta} + \mu\, \overline{B}(\tau), \tag{25}$$

where μ is a constant, which must be chosen in a way to reduce the right hand side of (25) to $\overline{B}(\tau)$. Multiplying both sides of (25) by $\exp(-\tau\eta)$ and integrating we find

$$\int_0^\infty \exp(-\eta\tau)\, B'(\tau)\, d\tau = 2\frac{B(0)}{\eta} \int_1^\infty r(\eta, \zeta)\, \frac{d\zeta}{\zeta} + \frac{\mu}{\eta}\, r(\eta),$$

where $r(\eta, \zeta)$ is the function introduced in earlier paragraphs. Integrating by parts in the left hand side of this equation we find

$$\eta \int_0^\infty \exp\left(-\eta\tau\right) B(\tau)\, d\tau = \overline{B}(0) \left[1 + \frac{2}{\eta} \int_1^\infty r(\eta, \zeta)\, \frac{d\zeta}{\zeta}\right] + \frac{\mu}{\eta}\, r(\eta),$$

or

$$r(\eta) \left(1 - \frac{\mu}{\eta}\right) = \overline{B}(0) \left[1 + 2 \int_1^\infty R(\eta, \zeta)\, \frac{d\zeta}{\zeta}\right]. \tag{26}$$

Substituting here instead of $R(\eta, \zeta)$ its expression, and writing y instead of η^{-1} and z instead of ζ^{-1} we obtain

$$r(y) = \frac{\overline{B}(0)}{1 - \mu y} \left[1 + 2y\varphi(y) \int_0^1 \frac{\varphi(z)\, dz}{z + y}\right], \tag{27}$$

with $\varphi(z)$ as defined in §2 (see also the Table 2). Taking into account (12) we can rewrite (27) in the form

$$r(y) = \frac{2}{\sqrt{\lambda}}\, \frac{\overline{B}(0)\, \varphi(y)}{1 - \mu y}. \tag{28}$$

We see that the intensity of radiation which leaves the medium under the angle $\arccos \varphi(y)$ is proportional to

$$\frac{\varphi(y)}{1 - \mu y}.$$

Let us now determine the parameter μ. From the equation (21) we have

$$\overline{B}(0) = \frac{\lambda}{2} \int_0^\infty \mathrm{Ei}t\, B(t)\, dt,$$

or

$$\overline{B}(0) = \frac{\lambda}{2} \int_0^1 \frac{d\zeta}{\zeta} \int_0^\infty \exp\left(-t\zeta\right) \overline{B}(t)\, dt = \frac{\lambda}{2} \int_1^\infty \frac{r(\zeta)}{\zeta^2}\, d\zeta = \frac{\lambda}{2} \int_0^1 r(y)\, dy.$$

Taking for $r(y)$ the expression (28) we find an equation from which μ can be determined as a function of λ.

$$\sqrt{\lambda} \int_0^1 \frac{\varphi(y)\, dy}{1 - \mu y} = 1. \tag{29}$$

Let us now show that μ is the solution of

$$\lambda = \frac{2\mu}{\ln \frac{1+\mu}{1-\mu}}. \tag{30}$$

To prove this we consider the bounded solution of the equation

$$\frac{\lambda}{2} \int_1^\infty \exp\left(-\tau\xi\right) \frac{d\xi}{\xi\xi - \mu)} = C(\tau) - \frac{\lambda}{2} \int_0^\infty \mathrm{Ei}|\tau - t, C(t)\, dt, \tag{31}$$

where $\mu > 0$ satisfies (30). The bounded solution of (31) can be expressed as superposition of solutions of (3). Hence

$$C(\tau) = 2\int_1^\infty \frac{B(\tau,\xi)\,d\xi}{\xi(\xi-\mu)}. \tag{32}$$

Multiplying this equation by $\exp(-\eta\tau)$ and integrating we find

$$\int_0^\infty C(\tau)\exp(-\eta\tau)\,d\tau = 2\int_1^\infty \frac{R(\eta,\xi)\,d\xi}{\xi(\xi-\mu)}. \tag{33}$$

On the other hand we can write explicitly the only bounded solution of (31) which is

$$C(\tau) = \exp(-\mu\tau). \tag{34}$$

Therefore we rewrite (33) in the form

$$\frac{1}{\mu+\eta} = 2\int_1^\infty \frac{\varphi(y)\,\varphi(\xi)\,d\xi}{\xi(\xi+\eta)(\xi-\mu)}.$$

Substituting again $\eta^{-1} = y$ and $\xi^{-1} = x$ we find

$$\frac{1}{1+\mu y} = 2\varphi(y)\int_0^1 \frac{x\,\varphi(x)\,dx}{(x+y)(1-\mu x)}, \tag{35}$$

or integrating both sides of (35)

$$\frac{1}{\mu}\ln(1+\mu) = 2\int_0^1 \frac{x\varphi(x)\,dx}{1-\mu x}\int_0^1 \frac{\varphi(y)\,dy}{x+y}.$$

By virtue of the functional equation (12)

$$\frac{1}{\mu}\ln(1+\mu) = \int_0^1 \frac{\frac{2}{\sqrt{\lambda}}\varphi(x)-1}{1-\mu x}\,dx.$$

This yields

$$\frac{1}{\mu}\ln\frac{1+\mu}{1-\mu} = \frac{2}{\lambda}\int_0^1 \frac{\varphi(x)\,dx}{1-\mu x},$$

or on the basis of (30)

$$\sqrt{\lambda} = \int_0^1 \frac{\varphi(x)\,dx}{1-\mu x}.$$

This means that the root of the equation (29) actually is the quantity μ determined by (25). In the special case of $\lambda = 1$, taking into account (28) we have

$$r(y) = A\,\varphi(y),$$

where A is a constant. Since the mathematical problem of distribution of brightness over solar disk is equivalent to the problem considered in this paragraph we can conclude that $\varphi(y)$ is the function representing the distribution of brightness over the solar disk for $\lambda = 1$.

§7. Conclusions

Our method of reduction of an integral equation to a fuunctional equation is applicable not only to the equation of scattering theory but also to more general equations with kernels depending on the difference of variables. The whole argumentation remains the same.

In another paper we intend to apply our method to the case of nonspherical indicatrix of scattering.

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